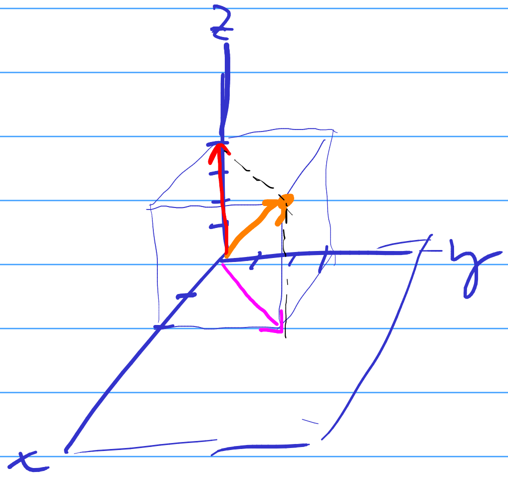
Úloha 1. V $\mathbb{R}^3$ je dán vektor $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$. Spočtěte jeho kolmou projekci na osu z (tedy na přímku $\mathsf{span}(\mathbf{e}_3)$) a na rovinu xy (tedy na rovinu $\mathsf{span}(\mathbf{e}_1, \mathbf{e}_2)$). Popište matice projekce na osu z a na rovinu xy.

$$\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathrm{span}(\vec{e}_1, \vec{e}_2)$$

$$\mathrm{proj}_z(\vec{b}) = ? = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \qquad \mathrm{proj}_{xy}(\vec{b}) = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$



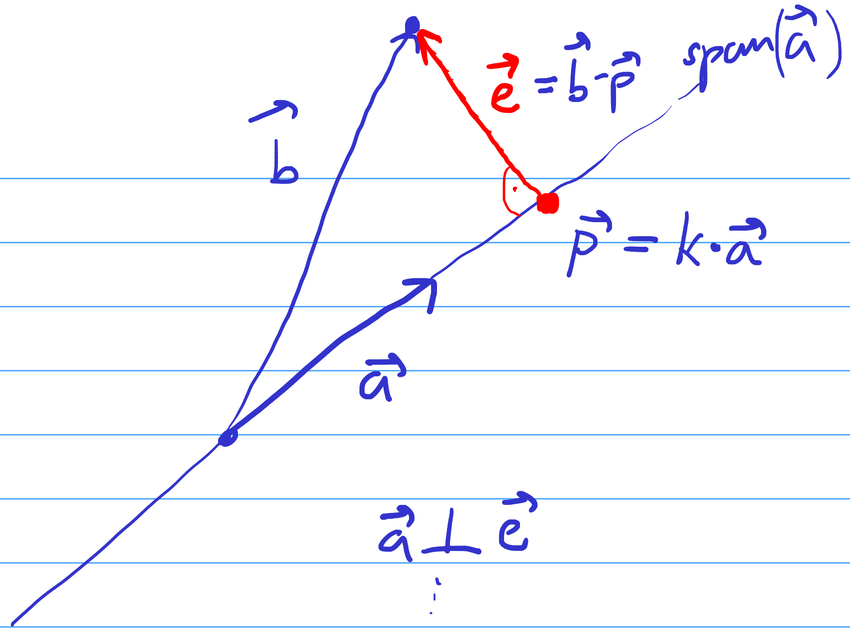
$$\mathbb{P}_{xy} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{P}_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Úloha 2. V $\mathbb{R}^n$ je dán nenulový vektor

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Odvoďte vzorec pro výpočet kolmé projekce na přímku $\mathsf{span}(\mathbf{a})$. To jest, máme-li dán vektor $\mathbf{b} \in \mathbb{R}^n$, odvoďte, jak spočítat $\mathrm{proj}_{\mathsf{span}(\mathbf{a})}(\mathbf{b})$.



$$\vec{a} \perp \vec{e}$$

$$\langle \vec{a} | \vec{e} \rangle = 0$$

$$\langle \vec{a} | \vec{e} \rangle = \langle \vec{a} | \vec{b} - \vec{p} \rangle = \underbrace{\langle \vec{a} | \vec{b} - k \cdot \vec{a} \rangle = 0}.$$

$$\langle \vec{a} | \vec{b} \rangle - \langle \vec{a} | k \cdot \vec{a} \rangle = 0$$

$$\langle \vec{a} | \vec{b} \rangle - k \cdot \langle \vec{a} | \vec{a} \rangle = 0$$

$$\boxed{k = \frac{\langle \vec{a} | \vec{b} \rangle}{\langle \vec{a} | \vec{a} \rangle}}$$

$$\boxed{\vec{p} = k \cdot \vec{a} = \frac{\langle \vec{a} | \vec{b} \rangle}{\langle \vec{a} | \vec{a} \rangle} \cdot \vec{a}.}$$

Úloha 3. 1. Spočtěte kolmou projekci vektoru $\mathbf{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ na přímku $\mathsf{span}(\mathbf{a})$, kde $\mathbf{a} = \begin{pmatrix}1\\2\\2\end{pmatrix}$.

2. Nalezněte matici $\mathbf{P}$ kolmé projekce na přímku $\mathsf{span}(\mathbf{a})$.

3. Spočtěte $\mathbf{P} \cdot \mathbf{P}$.

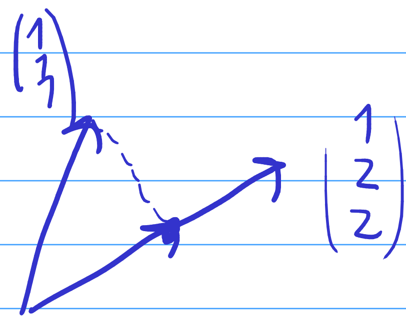
4. Jak vypadá matice kolmé projekce na přímku $\mathsf{span}(2 \cdot \mathbf{a})$?

$$\vec{p} = \frac{\langle \vec{a} | \vec{b} \rangle}{\langle \vec{a} | \vec{a} \rangle} \cdot \vec{a}$$

1.) Jsme v $\mathbb{R}^3$ (se std. sk. součinem)

$$\vec{p} = \mathrm{proj}_{\mathrm{span}(\vec{a})}(\vec{b}) = \frac{\left\langle \begin{pmatrix}1\\2\\2\end{pmatrix} \middle| \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\2\\2\end{pmatrix} \middle| \begin{pmatrix}1\\2\\2\end{pmatrix} \right\rangle} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{5}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\left\langle \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \middle| \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$



1. sloupec matice P

$$\vec{p}_1 = \mathrm{proj}_{\mathrm{span}(\vec{a})} \overbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}^{\vec{e}_1} = \frac{\overbrace{\left\langle \vec{a} \middle| \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle}^{1}}{\underbrace{\langle \vec{a} | \vec{a} \rangle}_{9}} \cdot \vec{a} = \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\vec{p}_2 = \mathrm{proj}_{\mathrm{span}(\vec{a})} \begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{2}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\vec{p}_3 = \ldots = \frac{2}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

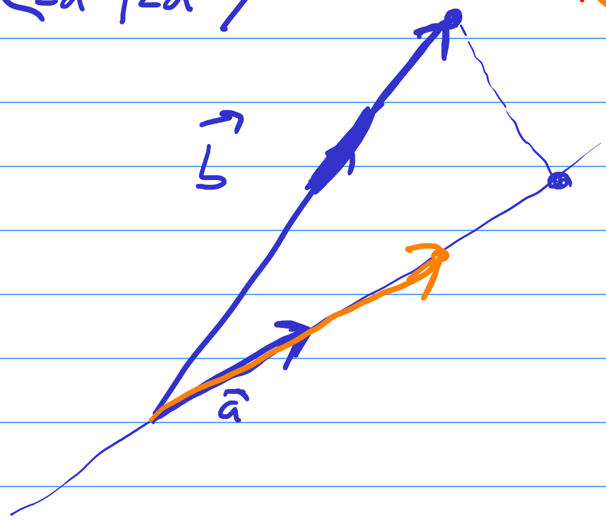
$$\boxed{P = \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix}}$$

3

$$\underline{P \cdot P} = \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix} = \frac{1}{9^2}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix}$$

$$= \frac{1}{9^2}\begin{pmatrix}9 & 18 & 18\\ 18 & 36 & 36\\ 18 & 36 & 36\end{pmatrix} = \frac{\cancel{9}}{\cancel{9^2}}\frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{pmatrix} = \underline{P}$$

$$\vec{p} = \frac{\langle 2\vec{a} \mid \vec{b} \rangle}{\langle 2\vec{a} \mid 2\vec{a} \rangle} \cdot 2\vec{a} = \cancel{2}\,\frac{\langle \vec{a} \mid \vec{b} \rangle}{\cancel{4}\langle \vec{a} \mid \vec{a} \rangle} \cdot \cancel{2}\vec{a} =$$



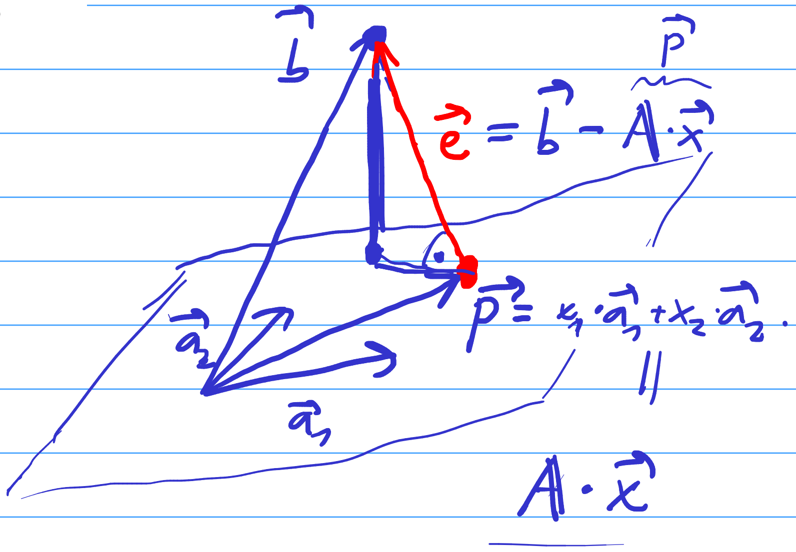
Úloha 4. V $\mathbb{R}^m$ je dán lineárně nezávislý seznam vektorů $(\mathbf{a}_1,\dots,\mathbf{a}_n)$. Odvoďte vzorec pro výpočet kolmé projekce na lineární podprostor $M = \mathsf{span}(\mathbf{a}_1,\dots,\mathbf{a}_n)$. To jest, máme-li dán vektor $\mathbf{b}\in\mathbb{R}^m$, odvoďte, jak spočítat $\mathrm{proj}_M(\mathbf{b})$.

$$\boxed{n=2}$$

$(\vec a_1, \vec a_2)$ LN

$M = \mathrm{span}(\vec a_1, \vec a_2)$

ať $A = \begin{pmatrix} | & | \\ \vec a_1 & \vec a_2 \\ | & | \end{pmatrix}$. $\quad \vec x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



$M \perp \vec e$

Chci zároveň: $\boxed{\begin{matrix} \vec a_1 \perp \vec e \\ \vec a_2 \perp \vec e \end{matrix}}$ $\quad \underbrace{\begin{matrix} \langle \vec a_1 | \vec e \rangle = 0 \\ \langle \vec a_2 | \vec e \rangle = 0 \end{matrix}}$

$$\boxed{\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = \vec x^T \cdot \vec y}$$

$$= x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$= (x_1 \ x_2 \ x_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\underbrace{\begin{matrix} \langle \vec a_1 | \vec b - A\vec x \rangle = 0 \\ \langle \vec a_2 | \vec b - A\vec x \rangle = 0 \end{matrix}}$$

$$\vec a_1^T \cdot (\vec b - A\vec x) = 0$$

$$\vec a_2^T \cdot (\vec b - A\vec x) = 0$$

$$\Updownarrow$$

$$\underbrace{\begin{pmatrix} - \vec a_1^T - \\ - \vec a_2^T - \end{pmatrix}}_{A^T} \cdot (\vec b - A\vec x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{A^T \cdot (\vec b - A\vec x) = \vec o}$$

$$A^T \vec b - A^T A \vec x = \vec o$$

$$\boxed{A^T A \vec x = A^T \vec b}$$

Úloha 6. Nechť $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ a $\mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$.

Nalezněte kolmou projekci vektoru $\mathbf{b}$ na lineární podprostor $\operatorname{im}(\mathbf{A})$.

$\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

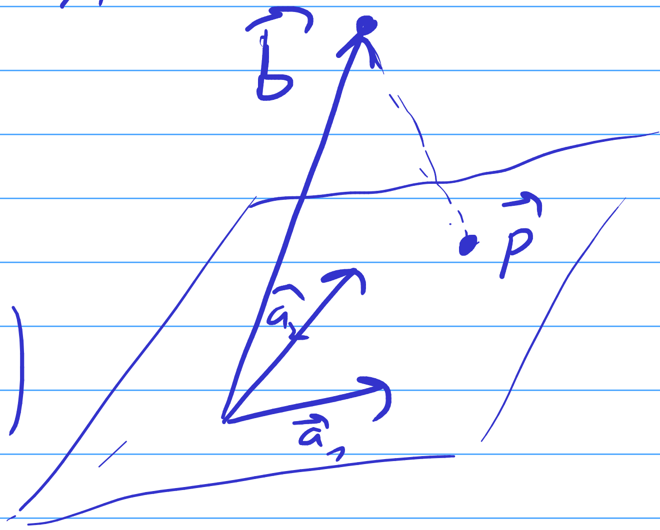
$\vec{p} = A \cdot \vec{x}$, kde $\vec{x}$ má splňovat rovnost

$$A^T \cdot A \cdot \vec{x} = A^T \cdot \vec{b}.$$

$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\qquad A^T \cdot A = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$; $A^T \cdot \vec{b} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

Řešíme soustavu $\left(\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right)$; řešením je $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

$\begin{vmatrix} 3 & 3 \\ 3 & 5 \end{vmatrix} = 6$ $\qquad \cancel{\begin{vmatrix} 3 & 6 \\ 3 & 0 \end{vmatrix}}$



$$\vec{p} = A \cdot \begin{pmatrix} 5 \\ -3 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}.$$

$\underline{A \cdot \vec{x} = \vec{b}}$ NŘ

$\downarrow$

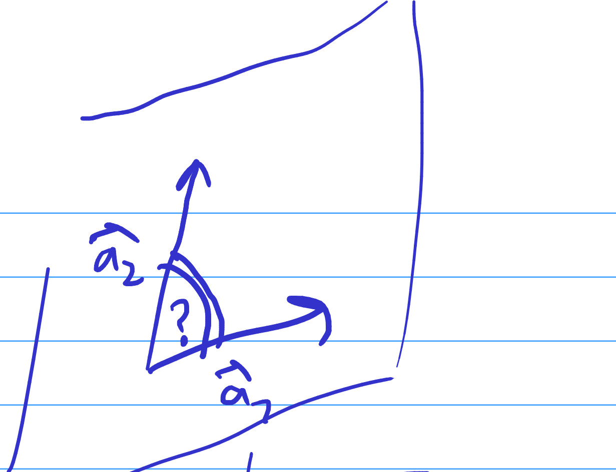
$A^T \cdot A \cdot \vec{x} = A^T \cdot \vec{b}$

$\vec{p} = A \cdot \vec{x} = \underline{A (A^T A)^{-1} A^T} \vec{b}$

Úloha 7. Ortogonalisujte následující seznamy vektorů:

1. $\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-2\end{pmatrix}, \begin{pmatrix}3\\-3\\3\end{pmatrix}\right)$ z $\mathbb{R}^3$.

$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3$



Vytvoříme seznam $Q = (\vec{q}_1, \vec{q}_2, \vec{q}_3)$

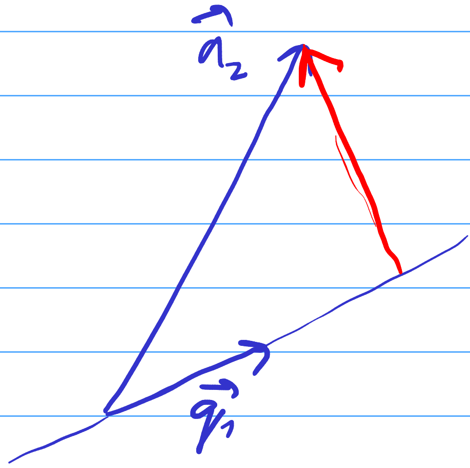
$$\vec{q}_1 = \vec{a}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\vec{q}_2 = \vec{a}_2 - \mathrm{proj}_{\mathrm{span}(\vec{q}_1)} \vec{a}_2 =$$

$$= \vec{a}_2 - \frac{\langle \vec{q}_1 | \vec{a}_2 \rangle}{\langle \vec{q}_1 | \vec{q}_1 \rangle} \cdot \vec{q}_1 =$$

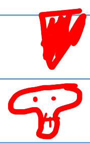
$$= \begin{pmatrix}2\\0\\-2\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$Q = \left(\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-2\end{pmatrix} \quad \right)$$



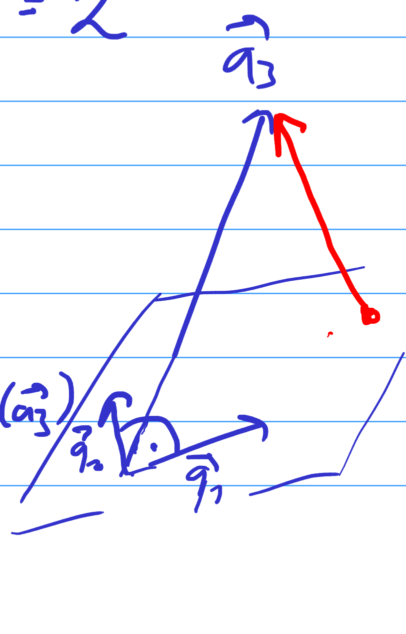
$$\langle \vec{q}_1 | \vec{q}_1 \rangle = 2$$

$$\vec{q}_3 = \vec{a}_3 - \mathrm{proj}_{\mathrm{span}(\vec{q}_1, \vec{q}_2)} (\vec{a}_3)$$



$\longrightarrow \; = \qquad \vec{q}_1 \perp \vec{q}_2$

$$= \vec{a}_3 - \mathrm{proj}_{\mathrm{span}(\vec{q}_1)} (a_3) - \mathrm{proj}_{\mathrm{span}(\vec{q}_2)} (\vec{a}_3)$$



Úloha 7. Ortogonalisujte následující seznamy vektorů:

1. $\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \right)$ z $\mathbb{R}^3$.

$\boxed{\boxed{\vec{a}_1 \quad \vec{a}_2} \quad \vec{a}_3}$

$Q = \boxed{\boxed{\vec{q}_1, \vec{q}_2}, \vec{q}_3}$

$\langle \vec{q}_2 | \vec{q}_2 \rangle = 6$

$Q = \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$

$\vec{q}_3 = \vec{a}_3 - \mathrm{proj}_{\mathrm{span}(\vec{q}_1, \vec{q}_2)} (\vec{a}_3)$

$\longrightarrow = \quad \vec{q}_1 \perp \vec{q}_2$

$= \vec{a}_3 - \mathrm{proj}_{\mathrm{span}(\vec{q}_1)} (a_3) - \mathrm{proj}_{\mathrm{span}(\vec{q}_2)} (\vec{a}_3)$

$= \vec{a}_3 - \dfrac{\langle \vec{q}_1 | \vec{a}_3 \rangle}{\langle \vec{q}_1 | \vec{q}_1 \rangle} \vec{q}_1 - \dfrac{\langle \vec{q}_2 | \vec{a}_3 \rangle}{\langle \vec{q}_2 | \vec{q}_2 \rangle} \vec{q}_2$

$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \dfrac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \dfrac{-6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Ortog.

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right) \rightsquigarrow \left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right)$$